DET KGL. DANSKE VIDENSKABERNES SELSKAB
MATEMATISK-FYSISKE MEDDELELSER, BIND XXV, NR. 20

A THEOREM ON THE MEAN MOTIONS OF ALMOST PERIODIC FUNCTIONS

BY

HANS TORNEHAVE

KØBENHAVN
I KOMMISSION HOS EJNAR MUNKSGAARD
1950

Printed in Denmark
Bianco Lunos Bogtrykkeri

INTRODUCTION

Let $f(t)$, $-\infty < t < \infty$ be a complex-valued almost periodic function satisfying a condition $|f(t)| \geq k > 0$, and let $\arg f(t)$ denote a continuous argument of $f(t)$. According to a theorem by H. BOHR[1] we have

$$\arg f(t) = ct + g(t)$$

where c is a real constant and $g(t)$ is an almost periodic function. The constant c is called the **mean motion** of $f(t)$.

The mean motion is unchanged if $f(t)$ is multiplied by an arbitrary complex constant. The function $\frac{1}{f(t)}$ will have the mean motion $-c$. If we have a relation $f(t) = f_1(t) f_2(t)$, where $f_1(t)$ and $f_2(t)$ are almost periodic functions satisfying the conditions $|f_1(t)| \geq k > 0$, $|f_2(t)| \geq k > 0$, then $c = c_1 + c_2$, where c_1 and c_2 are the mean motions of $f_1(t)$ and $f_2(t)$.

These trivial examples suggest that a relation between almost periodic functions $f_1(t), \cdots, f_n(t)$, where $|f_\nu(t)| \geq k > 0$, may imply a relation between the corresponding mean motions $c_1, \cdots, c_n$.

The first not trivial result in this direction was proved by B. JESSEN[2] and we shall state his theorem in the following form:

If $f_1(t)$ and $f_2(t)$ are almost periodic functions satisfying the conditions $|f_1(t)| \geq k > 0$, $|f_2(t)| \geq k > 0$, and further satisfying a linear relation

$$\alpha_1 f_1(t) + \alpha_2 f_2(t) = \beta,$$

[1] H. BOHR: Kleinere Beiträge zur Theorie der fastperiodischen Funktionen 1. D. Kgl. Danske Vidensk. Selskab, Mat.-fys. Medd. X 10 (1930). Über fastperiodische ebene Bewegungen. Comment. math. helv. 4 (1932).

[2] B. JESSEN: Über die Säkularkonstanten einer fastperiodischen Funktion. Math. Ann. 111 (1935).

where α_1 *and* α_2 *are* $\neq 0$, *then the corresponding mean motions* c_1 *and* c_2 *will satisfy a relation*

$$r_1 c_1 + r_2 c_2 = 0,$$

where r_1 *and* r_2 *are rational numbers and* $(r_1, r_2) \neq (0, 0)$.

The author[1] has recently proved the following generalization of Jessen's theorem:

If $f_1(t), \cdots, f_n(t)$ *are almost periodic functions satisfying the conditions* $|f_1(t)| \geqq k > 0, \cdots, |f_n(t)| \geqq k > 0$, *and further satisfying a linear relation*

$$\alpha_1 f_1(t) + \cdots + \alpha_n f_n(t) = \beta,$$

where $\alpha_1, \cdots, \alpha_n$ *are* $\neq 0$, *then the corresponding mean motions will satisfy a relation*

$$r_1 c_1 + \cdots + r_n c_n = 0,$$

where $r_1, \cdots, r_n$ *are rational numbers and* $(r_1, \cdots, r_n) \neq (0, \cdots, 0)$.

This theorem is an immediate corollary of the following result:

If $f_1(t), \cdots, f_n(t)$ *are almost periodic functions satisfying the conditions* $|f_1(t)| \geqq k > 0, \cdots, |f_n(t)| \geqq k > 0$, *and with rationally independent mean motions* $c_1, \cdots, c_n$ (i. e. with mean motions $c_1, \cdots, c_n$, which do not satisfy any linear relation $r_1 c_1 + \cdots + r_n c_n = 0$, where $r_1, \cdots, r_n$ are rational numbers and $(r_1, \cdots, r_n) \neq (0, \cdots, 0)$), *and if further* $v_1, \cdots, v_n$ *are arbitrary real numbers and* ε *is an arbitrary positive number, then there exists a real number* t^*, *such that we have simultaneously*

$$|\arg f_\nu(t^*) - v_\nu| \leqq \varepsilon; \quad \nu = 1, \cdots, n$$

for convenient choice of the arguments.

The author was further able to prove another generalization of Jessen's theorem concerning three almost periodic functions $f_1(t)$, $f_2(t)$, $f_3(t)$, satisfying a relation

$$\alpha_1 f_1^2 + \alpha_2 f_2^2 + \alpha_3 f_3^2 + \beta_1 f_2 f_3 + \beta_2 f_3 f_1 + \beta_3 f_1 f_2 = k.$$

Also in this case the mean motions will satisfy a linear relation $r_1 c_1 + r_2 c_2 + r_3 c_3 = 0$, and the theorem follows immediately from

[1] H. Tornehave: On a Generalization of Kronecker's Theorem. D. Kgl. Danske Vidensk. Selskab, Mat.-fys. Medd. XXIV 11 (1948).

the fact that it is possible to choose φ_1, φ_2 and φ_3 such that $\alpha_1 e^{2i\varphi_1}$, $\alpha_2 e^{2i\varphi_2}$, $\alpha_3 e^{2i\varphi_3}$, $\beta_1 e^{i(\varphi_2+\varphi_3)}$, $\beta_2 e^{i(\varphi_3+\varphi_1)}$ and $\beta_3 e^{i(\varphi_1+\varphi_2)}$ and $-k$ are situated in the same half-plane.

All theorems mentioned so far have been proved by an investigation of the arguments of almost periodic functions with rationally independent mean motions, and the behaviour of their absolute values has not been taken into account. The absolute values, however, are also subject to quite important restrictions, and one might therefore expect that much stronger theorems would hold. In the present paper we shall prove a generalization of this type, dealing with more general relations between two almost periodic functions. Our principal result will be the following theorem.

Theorem 1. *Let $F(z_1, z_2)$ be an integral function, which is not identically zero. If $f_1(t)$ and $f_2(t)$ are almost periodic functions satisfying the conditions $|f_1(t)| \geq k > 0$, $|f_2(t)| \geq k > 0$, and further satisfying the relation*

$$F(f_1(t), f_2(t)) = 0, \; -\infty < t < \infty,$$

then the corresponding mean motions c_1 and c_2 will satisfy a relation

$$r_1 c_1 + r_2 c_2 = 0 \tag{1}$$

where r_1 and r_2 are rational numbers and $(r_1, r_2) = (0, 0)$.

A careful examination of our proof will show that the function $F(z_1, z_2)$ need not be integral. Let M denote the closure of the set of all points $(f_1(t), f_2(t))$, $-\infty < t < \infty$, and let $U_1, \cdots, U_n$ be open domains in the (z_1, z_2)-space, such that M is contained in $U_1 + \cdots + U_n$. Let $F_\nu(z_1, z_2)$; $\nu = 1, \cdots, n$ be analytic and not identically zero in U_ν. If then

$$F_\nu(f_1(t), f_2(t)) = 0; \quad \nu = 1, \cdots, n$$

for all values of t, for which $(f_1(t), f_2(t))$ belongs to U_ν, we have the relation (1).

The zeros of $F(z_1, z_2)$ will form a two-dimensional manifold S in four-dimensional real space, and we may consider $z_1 = f_1(t)$, $z_2 = f_2(t)$ as an almost periodic movement on S. Almost periodic movements on two-dimensional surfaces were studied by FENCHEL and JESSEN[1], who proved that an almost periodic movement on a two-dimensional surface admitting the non-Euclidean plane as universal covering can be transformed continuously into a periodic movement. If the manifold of zeros of the function $F(z_1, z_2)$ of Theorem 1 is a surface of this kind, Theorem 1 will follow from Fenchel's and Jessen's theorem. The manifold of zeros may, however, consist of several surfaces intersecting each other, and the theorem of Fenchel and Jessen can not be applied in that case. We are going to prove Theorem 1 by a method very different from the one applied by Fenchel and Jessen; in fact, our proof will be based on the analytic properties of the manifold of zeros, while the proof of Fenchel and Jessen was based on topological properties of the surfaces.

Our proof will be indirect. We shall assume from the start that Theorem 1 is wrong and most of our considerations will therefore be based on the following (false) assumption:

A. *There exist an integral function $F(z_1, z_2)$, which is not identically zero, and two almost periodic functions $f_1(t)$ and $f_2(t)$ satisfying the conditions $|f_1(t)| \geqq k > 0$, $|f_2(t)| \geqq k > 0$. The mean motions c_1 and c_2 of $f_1(t)$ and $f_2(t)$ are rationally independent and we have identically*

$$F(f_1(t),\ f_2(t)) = 0.$$

In § 1 we shall bring the assumption **A** on a more convenient form. For this purpose we need a few well-known results concerning almost periodic functions. These results are briefly outlined in the earlier paper by the author, quoted above[2]. In § 2 we shall prove that the functions $f_1(t)$ and $f_2(t)$ in the assumption **A** can be replaced by differentiable functions. The proof of Theorem 1 follows in § 3 and § 4. Our proof will be based on the special properties of analytic manifolds in four-dimensional space, and it admits no obvious generalizations to the case of relations involving more than two almost periodic functions.

[1] W. FENCHEL und B. JESSEN: Über fastperiodische Bewegungen in ebenen Bereichen und auf Flächen. D. Kgl. Danske Vidensk. Selskab, Mat.-fys. Medd. XIII 6 (1935).

[2] H. TORNEHAVE, loc. cit.

§ 1.

An almost periodic function $f(t)$ can be written on the form

$$f(t) = \Phi(\beta_1 t, \beta_2 t, \cdots),$$

where $\beta_1, \beta_2, \cdots$ are real numbers satisfying the condition that $\beta_1, \cdots, \beta_n$ are rationally independent for every value of n, while $\Phi(x_1, x_2, \cdots)$ is a so-called limit periodic function with limit period 2π, i. e. to $\varepsilon > 0$ corresponds a continuous function $P(x_1, \cdots, x_n)$ periodic in each variable with a period which is an integral multiple of 2π, such that $|\Phi(x_1, x_2, \cdots) - P(x_1, \cdots, x_n)| \leq \varepsilon$ for all values of $x_1\, x_2, \cdots$. The function $\Phi(x_1, x_2, \cdots)$ is called a spatial extension of $f(t)$, and it is uniquely determined by $f(t)$ and the basis $\beta_1, \beta_2, \cdots$. A finite number of almost periodic functions will possess a common basis, and this basis can be chosen such that it contains a previously given sequence of rationally independent numbers. Let us choose a common basis $\beta_1, \beta_2, \cdots$ for the functions $f_1(t)$ and $f_2(t)$ of the assumption A, and let us choose $\beta_1 = c_1$, $\beta_2 = c_2$. Let $\Phi_1(x_1, x_2, \cdots)$ and $\Phi_2(x_1, x_2, \cdots)$ be the spatial extensions. We have

$$\arg \Phi_1(x_1, x_2, \cdots) = x_1 + \Psi_1(x_1, x_2, \cdots)$$
$$\arg \Phi_2(x_1, x_2, \cdots) = x_2 + \Psi_2(x_1, x_2, \cdots),$$

where $\Psi_1(x_1, x_2, \cdots)$ and $\Psi_2(x_1, x_2, \cdots)$ are limit periodic functions with limit period 2π. The relation $F(f_1(t), f_2(t)) = 0$ implies

$$F(\Phi_1(x_1, x_2, \cdots), \Phi_2(x_1, x_2, \cdots)) = 0 \tag{2}$$

for all values of $x_1, x_2, \cdots$.

To bring this relation in a more convenient form we introduce the analytic function

$$H(s_1, s_2) = H(\sigma_1 + it_1, \sigma_2 + it_2) = F(e^{s_1}, e^{s_2})$$

which is periodic with period $2\pi i$ in each variable. If we further put

$$\varphi_1(x_1, x_2) = \log |\Phi_1(x_1, x_2, 0, 0, \cdots)|$$
$$\varphi_2(x_1, x_2) = \log |\Phi_2(x_1, x_2, 0, 0, \cdots)|$$
$$\psi_1(x_1, x_2) = \Psi_1(x_1, x_2, 0, 0, \cdots)$$
$$\psi_2(x_1, x_2) = \Psi_2(x_1, x_2, 0, 0, \cdots)$$

the relation (2) takes the form

$$(3)\quad H(\varphi_1(x_1, x_2)+i(x_1+\psi_1(x_1, x_2)), \varphi_2(x_1, x_2)+i(x_2+\psi_2(x_1, x_2))) = 0$$

in other words, the surface with the parametric representation

$$(4)\quad \begin{cases} \sigma_1 = \varphi_1(x_1, x_2); & t_1 = x_1+\psi_1(x_1, x_2) \\ \sigma_2 = \varphi_2(x_1, x_2); & t_2 = x_2+\psi_2(x_1, x_2) \end{cases}$$

is situated on the manifold of zeros of the analytic function $H(s_1, s_2)$. It is important for our proof that the four functions $\varphi_1(x_1, x_2)$, $\varphi_2(x_1, x_2)$, $\psi_1(x_1, x_2)$ and $\psi_2(x_1, x_2)$ have the limit period 2π. Hence these four functions are continuous and to $\varepsilon > 0$ corresponds an integer N, such that we have

$$|\varphi_1(x_1+2\pi\nu N, x_2)-\varphi_1(x_1, x_2)| \leqq \varepsilon$$
$$|\varphi_1(x_1, x_2+2\pi\nu N)-\varphi_1(x_1, x_2)| \leqq \varepsilon$$

for all real values of x_1 and x_2 and for all integral values of ν, and the same condition is satisfied by the functions $\varphi_2(x_1, x_2)$, $\psi_1(x_1, x_2)$ and $\psi_2(x_1, x_2)$.

§ 2.

Let ε be a given positive number. It is easy to prove the existence of a function $\varphi_1^*(x_1, x_2)$ with limit period 2π in each variable and satisfying the condition

$$|\varphi_1(x_1, x_2)-\varphi_1^*(x_1, x_2)| \leqq \varepsilon$$

for all real values of x_1 and x_2 and we can further choose $\varphi_1^*(x_1, x_2)$ such that its partial derivatives are bounded and continuous. We may e. g. define $\varphi_1^*(x_1, x_2)$ as a convenient Féjér sum of the Fourier series of $\varphi_1(x_1, x_2)$. If, however, we apply this process on each of the functions φ_1, φ_2, ψ_1 and ψ_2, the surface

$$\sigma_1 = \varphi_1^*(x_1, x_2);\quad t_1 = x_1+\psi_1^*(x_1, x_2)$$
$$\sigma_2 = \varphi_2^*(x_1, x_2);\quad t_2 = x_2+\psi_2^*(x_1, x_2)$$

will not always be situated on the manifold of zeros of $H(s_1, s_2)$. We must therefore choose $\varphi_1^*(x_1, x_2)$ and $\psi_1^*(x_1, x_2)$ and afterwards determine $\varphi_2^*(x_1, x_2)$ and $\psi_2^*(x_1, x_2)$ such that (3) still holds, i. e. such that

$$(5)\quad H(\varphi_1^*+i(x_1+\psi_1^*), \varphi_2^*+i(x_2+\psi_2^*)) = 0.$$

We can change φ_1 and ψ_1 into φ_1^* and ψ_1^* by a continuous transformation, e.g. by means of the continuous families $\varphi_t = (1-t)\varphi_1 + t\varphi_1^*$ and $\psi_t = (1-t)\psi_1 + t\psi_1^*$ and we may choose $(s_1(t), s_2(t))$ on the manifold of zeros of $H(s_1, s_2)$ such that the functions $s_1(t)$ and $s_2(t)$ are continuous for fixed (x_1, x_2), and such that $s_1(t) = \varphi_t + i(x_1 + \psi_t)$. If $(s_1(t), s_2(t))$ never passes through a singular point on the manifold of zeros, we may put $s_2(1) = \varphi_2^* + i(x_2 + \psi_2^*)$ and the functions φ_2^* and ψ_2^* thus obtained will be continuos and differentiable and satisfy (5). The trouble is that the manifold of zeros usually possesses some singularities and the curve $(s_1(t), s_2(t))$ must occasionally pass through a singular point. We must therefore more carefully study the singularities on the manifold of zeros and take them into account in our considerations.

Let (a_1, a_2) be a zero of $H(s_1, s_2)$. In a certain neighbourhood of (a_1, a_2) we have according to Weierstrass' preparation theorem

$$H(s_1, s_2) = H^*(s_1, s_2)\, P(s_1, s_2),$$

where $H^*(s_1, s_2)$ is an analytic function without zeros, while $P(s_1, s_2)$ is a so-called pseudopolynomial, i. e. a function of the form

$$P(s_1, s_2) = A_0(s_1)(s_2 - a_2)^p + A_1(s_1)(s_2 - a_2)^{p-1} + \cdots + A_p(s_1).$$

By convenient choice of $H^*(s_1, s_2)$ we have $A_0(s_1) = (s_1 - a_1)^q$, and for $\nu \geqq 1$ we have $A_\nu(s_1) = A_\nu^*(s_1)(s_1 - a_1)^q$, where $A_\nu^*(s_1)$ is analytic in the neighbourhood of a_1 and $A_\nu^*(a_1) = 0$.

The pseudopolynomial $P(s_1, s_2)$ has coefficients from the integral domain of functions of one variable, analytic in a neighbourhood of a_1. An element $u(s_1)$ of this integral domain is a unity, if and only if $u(a_1) \neq 0$, and the only prime elements are the functions $u(s_1)(s_1 - a_1)$. Every function of the integral domain can in a certain neighbourhood of a_1 be written on the form $u(s_1)(s_1 - a_1)^p$ and this representation is unique. But we have then according to a well-known theorem from the algebra a unique factorization also in the integral domain of pseudopolynomials, i. e. every pseudopolynomial can in exactly one way be written as a product

$$P(s_1, s_2) = (s_1 - a_1)^q (P_1(s_1, s_2))^{q_1} \cdots (P_l(s_1, s_2))^{q_l} \tag{6}$$

where $P_1(s_1, s_2), \cdots, P_l(s_1, s_2)$ are irreducible pseudopolynomials. The representation (6) holds in a certain bicylinder

$$(7) \qquad |s_1 - a_1| \leqq \varrho; \quad |s_2 - a_2| \leqq \varrho.$$

The zero (a_1, a_2) is called a normal zero of $H(s_1, s_2)$, if all factors of the product (6) are identical, i. e. if $P(s_1, s_2)$ is a power of some irreducible pseudopolynomial. The zeros, which are not normal, are called critical. We shall prove that the number of critical zeros in the bicylinder (7) is finite. Let $R_{\mu\nu}(s_1)$ denote the resultant of $P_\mu(s_1, s_2)$ and $P_\nu(s_1, s_2)$, when $\mu \neq \nu$, and let $R_{\nu\nu}(s_1)$ denote the discriminant of $P_\nu(s_1, s_2)$. As the pseudopolynomials $P_\nu(s_1, s_2)$ are irreducible and different, the functions $R_{\mu\nu}(s_1)$ are analytic and not identically zero. If a zero (b_1, b_2), where $b_1 \neq a_1$, is critical, then the product $\prod_{\mu,\nu=1}^{l} R_{\mu\nu}(s_1)$ must vanish, when $s_1 = b_1$. This, however, will happen only for a finite set of values of b_1. Obviously the zeros (a_1, b_2), where $b_2 \neq a_2$, are normal. Hence the number of critical zeros inside the bicylinder (7) is finite and it follows that the critical zeros are isolated. In the following we shall consider $H(s_1, s_2)$ only inside a certain fixed tube $\sigma_1^2 + \sigma_2^2 \leqq C^2$ and, taking into account that $H(s_1, s_2)$ is periodic, we conclude that there exists a positive constant d, such that the distance between two arbitrary critical points is $\leqq d$.

Accordingly we may arrange all critical points on the manifold of zeros of $H(s_1, s_2)$ inside a fixed tube which contains the surface (4) in a sequence $\left(a_1^{(\nu)}, a_2^{(\nu)}\right)$; $\nu = 1, 2, \cdots$. Let η be a positive number, and let U_ν denote a neighbourhood of $\left(a_1^{(\nu)}, a_2^{(\nu)}\right)$ consisting of all points on the manifold of zeros of $H(s_1, s_2)$ that satisfies

$$0 > |s_1 - a_2^{(\nu)}| > \eta$$

and all points on the manifold of zeros of $H(s_1, s_2)$ that satisfies

$$|s_2 - a_2^{(\nu)}| < \eta; \; s_1 = a_1^{(\nu)}.$$

From the representation $P(s_1, s_2) = 0$ of the manifold of zeros follows that we can choose η so small that the distance from $\left(a_1^{(\nu)}, a_2^{(\nu)}\right)$ to an arbitrary point of the neighbourhood U_ν is $< \frac{\varepsilon}{3} < \frac{d}{3}$ and, as $H(s_1, s_2)$ is periodic, we can even choose η so small that this condition holds simultaneously for all values of ν.

Let M_ν denote the set of points in the (x_1, x_2)-plane, for which

$$s_1 = \varphi_1(x_1, x_2) + i(x_1 + \psi_1(x_1, x_2));\quad s_2 = \varphi_2(x_1, x_2) + i(x_2 + \psi_2(x_1, x_2))$$

is a point of the neighbourhood U_ν. As φ_1, φ_2, ψ_1 and ψ_2 are continuous there exists a number Δ, such that the distance between M_μ and M_ν is $\geqq \Delta$, when μ and ν are arbitrary positive integers. The complementary set of $M_1 + M_2 + \cdots$ is divided in two open sets N_1 and N_2, where N_1 consists of points (x_1, x_2) corresponding to points (s_1, s_2) in the neighbourhood of which the manifold of zeros of $H(s_1, s_2)$ has the form $s_2 = h(s_1)$, while N_2 consists of points (x_1, x_2) corresponding to points (s_1, s_2) in the neighbourhood of which the manifold of zeros has the form $s_1 =$ constant. The sets N_1 and N_2 are separated by $M_1 + M_2 + \cdots$, but they need not be connected and one of them may be empty.

We choose a positive number $\delta < \frac{d}{3\sqrt{2}}$, such that the oscillation of each of the functions $\varphi_1 + i(x_1 + \psi_1)$ and $\varphi_2 + i(x_2 + \psi_2)$ is $\leqq \frac{\varepsilon}{2\sqrt{2}}$ on every square with its side $\leqq \delta$. We divide the (x_1, x_2)-plane in squares with side δ by means of straight lines parallel to the axes. Let M'_ν denote the sum of all squares that contain a point of M_ν or its boundary in their interior. The boundary of each set M'_ν consists of a number of closed broken lines, but we replace it by a curve differentiable an infinity of times by replacing each corner on the boundary by a small arc similar to the arc

$$e^{-\frac{1}{x_1^2}} + e^{-\frac{1}{x_2^2}} = 1;\quad x_1 \geqq 0,\; x_2 \geqq 0.$$

If the corner is convex, we choose the arc so small that no part of M_ν is cut away, but if the corner is concave, we let the arc join the midpoints of the adjoining square. A corner, where two squares of M'_ν have one common angle is treated as two concave corners. If we count the curvature positive, when the curve is concave, the new boundary will possess a finite maximal curvature corresponding to a definite radius of curvature ϱ. Let M^*_ν denote the set, into which M'_ν is changed by our modification of the boundary, and let M^{**}_ν denote the set of all points with distance $\leqq \frac{\varrho}{2}$ from M^*_ν. The set M^{**}_ν is bounded by a curve parallel to the boundary of M_ν^* and differentiable everywhere an infinity of times. Let N_1^* and N_2^* denote the parts of N_1 and N_2 outside M_ν^{**}.

Let $S_1(x_1, x_2)$ and $S_2(x_1, x_2)$ denote exponential sums satisfying

$$(8)\quad \begin{cases} |\varphi_1(x_1, x_2)+i\psi_1(x_1, x_2)-S_1(x_1, x_2)| \leq \operatorname{Min}\left(\dfrac{\varepsilon}{6\sqrt{2}}, \dfrac{\eta}{2}\right) \\ |\varphi_2(x_1, x_2)+i\psi_2(x_1, x_2)-S_2(x_1, x_2)| \leq \operatorname{Min}\left(\dfrac{\varepsilon}{6\sqrt{2}}, \dfrac{\eta}{2}\right). \end{cases}$$

We define

$$(9)\quad \begin{cases} \varphi_1^*(x_1, x_2)+i\psi_1^*(x_1, x_2) = S_1(x_1, x_2) & \text{in } N_1^* \\ \varphi_2^*(x_1, x_2)+i\psi_2^*(x_1, x_2) = S_2(x_1, x_2) & \text{in } N_2^* \\ \left.\begin{aligned} \varphi_1^*(x_1, x_2)+i(x_1+i\psi_1^*(x_1, x_2)) &= a_1^{(\nu)} \\ \varphi_2^*(x_1, x_2)+i(x_2+i\psi_2^*(x_1, x_2)) &= a_2^{(\nu)} \end{aligned}\right\} & \text{in } M_\nu^*. \end{cases}$$

If we put $(0 \leq t \leq 1)$

$$(10)\quad \varphi_1^{(t)}+i\psi_1^{(t)} = (1-t)(\varphi_1+i\psi_1)+t(\varphi_1^*+i\psi_1^*) \quad \text{in } N_1^*$$

there exists a uniquely determined function

$$\varphi_2^{(t)}(x_1, x_2)+i\psi_2^{(t)}(x_1, x_2),$$

continuous as a function of t, x_1 and x_2, when $0 \leq t \leq 1$ and (x_1, x_2) belongs to N_1^* and satisfying

$$H\left(\varphi_1^{(t)}+i(x_1+\psi_1^{(t)}),\ \varphi_2^{(t)}+i(x_2+\psi_2^{(t)})\right) = 0$$

and

$$\varphi_2^{(0)}+i\psi_2^{(0)} = \varphi_2+i\psi_2.$$

In fact, it follows from (8), (9) and (10) that

$$\left|\left(\varphi_1^{(t)}+i(x_1+\psi_1^{(t)})\right)-\left(\varphi_1+i(x_1+\psi_1)\right)\right| \leq \frac{\eta}{2}$$

and according to the definition of the neighbourhoods U_ν this implies that $\left(\varphi_1^{(t)}+i(x_1+\psi_1^{(t)}),\ \varphi_2^{(t)}+i(x_2+\psi_2^{(t)})\right)$ will never reach any of the critical points $\left(a_1^{(\nu)}, a_2^{(\nu)}\right)$. We define

$$\varphi_2^*(x_1, x_2)+i\psi_2^*(x_1, x_2) = \varphi_2^{(1)}(x_1, x_2)+i\psi_2^{(1)}(x_1, x_2) \quad \text{in } N_1^*.$$

We have

$$|(\varphi_2^* + i\psi_2^*) - (\varphi_2 + i\psi_2)| \leq K\frac{\varepsilon}{6},$$

where K is the maximum value of the derivative of the function $s_2 = h(s_1)$ representing the manifold of zeros at all points inside the tube and outside the neighbourhoods U_ν. According to the periodicity of $H(s_1, s_2)$ the constant K will be finite. We define

$$\varphi_1^*(x_1, x_2) + i\psi_1^*(x_1, x_2) = \varphi_1(x_1, x_2) + i\psi_1(x_1, x_2) \text{ in } N_2^*.$$

The function

$$E(t) = \frac{e^{-\frac{2t-1}{t(t-1)}}}{1 + e^{-\frac{2t-1}{t(t-1)}}}$$

is differentiable an infinity of times in the interval $0 \leq t \leq 1$, if we define $E(0) = 0$ and $E(1) = 1$, and we have

$$E^{(\nu)}(0) = E^{(\nu)}(1) = 0; \quad \nu = 1, 2 \cdots.$$

Let $r(x_1, x_2)$ denote the distance from (x_1, x_2) to the boundary of M_ν^*. We define

$$(11) \begin{cases} \varphi_1^*(x_1, x_2) + i(x_1 + \psi_1^*(x_1, x_2)) = \\ \quad = a_1^{(\nu)} + \left(S_1(x_1, x_2) + ix_1 - a_1^{(\nu)}\right) E\left(\dfrac{2\,r(x_1, x_2)}{\varrho}\right) \end{cases}$$

in all components of $M_\nu^{**} - M_\nu^*$, which separate M_ν^* and N_1^*, and

$$\varphi_2^*(x_1, x_2) + i(x_2 + \psi_2^*(x_1, x_2)) =$$

$$= a_2^{(\nu)} + \left(S_2(x_1, x_2) + ix_2 - a_2^{(\nu)}\right) E\left(\frac{2\,r(x_1, x_2)}{\varrho}\right);$$

$$\varphi_1^*(x_1, x_2) + i\psi_1^*(x_1, x_2) = \varphi_1(x_1, x_2) + i\psi_1(x_1, x_2)$$

in all components of $M_\nu^{**} - M_\nu^*$, which separate M_ν^* and M_2^*. From (11) and (8) follows that

$$\varphi_1^* + i(x_1 + \psi_2^*) \neq a_1^{(\nu)}$$

in the first kind of components of $M_\nu^{**} - M_\nu^*$, and we can therefore in exactly one way determine two continuous functions $\varphi_2^*(x_1, x_2)$ and $\psi_2^*(x_1, x_2)$ satisfying

$$H(\varphi_1^* + i(x_1 + \psi_1^*),\ \varphi_2^* + i(x_2 + \psi_2^*)) = 0.$$

We have now defined the functions $\varphi_1^*(x_1, x_2)$, $\psi_1^*(x_1, x_2)$, $\varphi_2^*(x_1, x_2)$ and $\psi_2^*(x_1, x_2)$ for all values of x_1 and x_2, and they are obviously differentiable an infinity of times everywhere. We cannot prove that they are limit periodic, but we can obviously choose the numbers η, ε and δ so small that the difference $|\varphi_1 - \varphi_1^*|$ and the analogous differences are smaller than $\frac{\varepsilon_0}{3}$, where ε_0 is a previously given positive number. There will then exist an integer N, such that we have

$$|\varphi_1^*(x_1 + 2\pi\nu N, x_2) - \varphi_1^*(x_1, x_2)| \leq \varepsilon_0$$
$$|\varphi_1^*(x_1, x_2 + 2\pi\nu N) - \varphi_1^*(x_1, x_2)| \leq \varepsilon_0$$

for all real values of x_1 and x_2 and all integral values of ν, and the three other functions will satisfy the analogous inequalities. It is further easy to prove that there exists a constant K_0, such that the four functions φ_1^*, ψ_1^*, φ_2^* and ψ_2^* and their partial derivatives of the first order are $\leq K_0$ in absolute value.

§ 3.

In the neighbourhood of every normal point of the manifold of zeros of $H(s_1, s_2)$ the manifold of zeros is a two-dimensional surface defined by an equation

$$s_2 = h(s_1) = h_1(\sigma_1, t_1) + i h_2(\sigma_1, t_1)$$

or by two real equations

$$(12) \quad \left\{ \begin{aligned} \sigma_2 &= h_1(\sigma_1, t_1) \\ t_2 &= h_2(\sigma_1, t_1) \end{aligned} \right.$$

where the functions h_1 and h_2 satisfy the Cauchy-Riemann differential equations

$$(13) \quad \left\{ \frac{\partial h_1}{\partial \sigma_1} = \frac{\partial h_2}{\partial t_1}, \ \frac{\partial h_1}{\partial t_1} = -\frac{\partial h_2}{\partial \sigma_1}. \right.$$

We can solve the equations (12) on the form

$$(14) \quad \left\{ \begin{aligned} t_1 &= U(\sigma_1, \sigma_2) \\ t_2 &= V(\sigma_1, \sigma_2), \end{aligned} \right.$$

if the condition

$$D = \begin{vmatrix} \frac{\partial h_1}{\partial t_1} & 0 \\ \frac{\partial h_2}{\partial t_1} & -1 \end{vmatrix} = -\frac{\partial h_1}{\partial t_1} = \frac{\partial h_2}{\partial \sigma_1} \neq 0$$

is satisfied.

Let us first assume that $\frac{\partial h_1}{\partial t_1}$ is not identically zero. The functional determinant D will then vanish only along a certain analytic curve on the surface $h_2 = h(s_1)$. In the neighbourhood of all other points of the surface $h_2 = h(s_1)$ we can represent this surface by the equations (14), and we have according to (12) and (13)

$$\frac{\partial h_1}{\partial \sigma_1} - \frac{\partial h_2}{\partial \sigma_1}\frac{\partial U}{\partial \sigma_1} = 0 \qquad \frac{\partial h_2}{\partial \sigma_1}\frac{\partial U}{\partial \sigma_2} - 1 = 0$$

$$\frac{\partial h_2}{\partial \sigma_1} + \frac{\partial h_1}{\partial \sigma_1}\frac{\partial U}{\partial \sigma_1} - \frac{\partial V}{\partial \sigma_1} = 0 \qquad \frac{\partial h_1}{\partial \sigma_1}\frac{\partial U}{\partial \sigma_2} - \frac{\partial V}{\partial \sigma_2} = 0,$$

hence

$$\frac{\partial U}{\partial \sigma_1} = \frac{1}{D}\frac{\partial h_1}{\partial \sigma_1} \qquad \frac{\partial U}{\partial \sigma_2} = -\frac{1}{D}$$

$$\frac{\partial V}{\partial \sigma_1} = \frac{1}{D}\left(\left(\frac{\partial h_1}{\partial \sigma_1}\right)^2 + \left(\frac{\partial h_2}{\partial \sigma_1}\right)^2\right) \qquad \frac{\partial V}{\partial \sigma_2} = -\frac{1}{D}\frac{\partial h_1}{\partial \sigma_1},$$

and it follows that

$$\frac{\partial U}{\partial \sigma_1} = -\frac{\partial V}{\partial \sigma_2}$$

and

$$\frac{\partial U}{\partial \sigma_1}\frac{\partial V}{\partial \sigma_2} - \frac{\partial U}{\partial \sigma_2}\frac{\partial V}{\partial \sigma_1} = 1.$$

It follows that the mapping (14) of the real plane on the imaginary plane leaves the area unchanged.

If D is identically zero, we have

$$\operatorname{Im}\frac{dh}{ds_1} = \frac{\partial h_2}{\partial \sigma_1} = 0,$$

hence

$$\frac{dh}{ds_1} = c,$$

where c is a real constant. In this case the surface of zeros is

$$s_2 = cs_1 + a,$$

where a is a complex number. The surface of zeros is a plane, perpendicular on both the real and the imaginary plane, and the projections on the real and the imaginary plane of an area situated in such a part of the manifold of zeros will have measures equal to zero.

If a bounded surface is situated on the manifold of zeros, it follows generally that the areas of its projections on the real and the imaginary plane are identical.

§ 4.

We shall now again consider the surface given by the parametric representation

$$(15)\quad \begin{cases} \sigma_1 = \varphi_1^*(x_1, x_2); & t_1 = x_1 + \psi_1^*(x_1, x_2) \\ \sigma_2 = \varphi_2^*(x_1, x_2); & t_2 = x_2 + \psi_2^*(x_1, x_2). \end{cases}$$

Let A denote a part of the surface (15), determined by the conditions

$$(16)\qquad 0 \leqq x_1 \leqq 2\pi N, \quad 0 \leqq x_2 \leqq 2\pi N.$$

The projection of A on the real plane will have the area

$$A_r = \int_0^{2\pi N}\int_0^{2\pi N}\left(\frac{\partial \varphi_1^*}{\partial x_1}\frac{\partial \varphi_2^*}{\partial x_2} - \frac{\partial \varphi_1^*}{\partial x_2}\frac{\partial \varphi_2^*}{\partial x_1}\right) dx_1\, dx_2$$

or, if B denotes the boundary of the rectangle (16), according to Gauss' formula

$$A_r = \int_B \varphi_1^*\left(\frac{\partial \varphi_2^*}{\partial x_1} dx_1 + \frac{\partial \varphi_2^*}{\partial x_2} dx_2\right) =$$

$$= \int_0^{2\pi N}\left(\varphi_1^*(2\pi N, x_2)\frac{d\varphi_2^*(2\pi N, x_2)}{dx_2} - \varphi_1^*(0, x_2)\frac{d\varphi_2^*(0, x_2)}{dx_2}\right) dx_2$$

$$- \int_0^{2\pi N}\left(\varphi_1^*(x_1, 2\pi N)\frac{d\varphi_2^*(x_1, 2\pi N)}{dx_1} - \varphi_1^*(x_1, 0)\frac{d\varphi_2^*(x_1, 0)}{dx_1}\right) dx_1.$$

Denoting these integrals A'_r and A''_r, we have

$$A'_r = \int_0^{2\pi N} (\varphi_1^*(2\pi N, x_2) - \varphi_1^*(0, x_2)) \frac{d\varphi_2^*(2\pi N, x_2)}{dx_2} dx_2$$

$$+ \int_0^{2\pi N} \varphi_1^*(0, x_2) \frac{d(\varphi_2^*(2\pi N, x_2) - \varphi_2^*(0, x_2))}{dx_2} dx_2$$

and we obtain by partial integration

$$\begin{aligned} A'_r = {} & \varphi_1^*(0, 2\pi N)\,(\varphi_2^*(2\pi N, 2\pi N) - \varphi_2^*(0, 2\pi N)) \\ & - \varphi_1^*(0, 0)\,(\varphi_2^*(2\pi N, 0) - \varphi_2^*(0, 0)) \\ & + \int_0^{2\pi N} (\varphi_1^*(2\pi N, x_2) - \varphi_1^*(0, x_2)) \frac{d\varphi_2^*(2\pi N, x_2)}{dx_2} dx_2 \\ & - \int_0^{2\pi N} (\varphi_2^*(2\pi N, x_2) - \varphi_2^*(0, x_2)) \frac{d\varphi_1^*(0, x_2)}{dx_2} dx_2, \end{aligned}$$

hence, if we according to the remarks at the end of § 2 choose $2\pi N$ as a translation number of φ_1^*, ψ_1^*, φ_2^* and ψ_2^*, we have

$$|A'_r| \leq 2K\varepsilon_0 + 4\pi N K_0 \varepsilon_0 .$$

We get a similar estimate for A''_r, so that we finally obtain

$$(17) \qquad |A_r| \leq 4K_0\varepsilon_0(2\pi N + 1) < 16\pi N K_0 \varepsilon_0 .$$

The projection of A on the imaginary plane will have the area

$$A_i = \int_B (x_1 + \psi_1^*) \left(\frac{\partial \psi_2^*}{\partial x_1} dx_1 + \left(1 + \frac{\partial \psi_2^*}{\partial x_2}\right) dx_2\right) =$$

$$= \int_B x_1 \, dx_2 + \int_B \psi_1^* \, dx_2 + \int_B x_1 \left(\frac{\partial \psi_2^*}{\partial x_1} dx_1 + \frac{\partial \psi_2^*}{\partial x_2} dx_2\right)$$

$$+ \int_B \psi_1^* \left(\frac{\partial \psi_2^*}{\partial x_1} dx_1 + \frac{\partial \psi_2^*}{\partial x_2} dx_2\right).$$

For the two first integrals we obtain immediately

$$\int_B x_1 dx_2 = 4\pi^2 N^2,$$

$$\left|\int_B \psi_1^* dx_2\right| \leq 4\pi K_0 N.$$

For the third integral we obtain

$$\int_B x_1 \left(\frac{\partial \psi_2^*}{\partial x_1} dx_1 + \frac{\partial \psi_2^*}{\partial x_2} dx_2\right) =$$

$$= \int_0^{2\pi N} x_1 \frac{\partial \psi_2^*(x_1, 0)}{dx_1} dx_1 + 2\pi N \int_0^{2\pi N} \frac{d\psi_2^*(2\pi N, x_2)}{dx_2} dx_2 - \int_0^{2\pi N} x_1 \frac{d\psi_2^*(x_1, 2\pi N)}{dx_1} dx_1 =$$

$$= 2\pi N(\psi_2^*(2\pi N, 0) - \psi_2^*(2\pi N, 2\pi N))$$
$$+ 2\pi N(\psi_2^*(2\pi N, 2\pi N) - \psi_2^*(2\pi N, 0))$$
$$+ \int_0^{2\pi N} (\psi_2^*(x_1, 2\pi N) - \psi_2^*(x_1, 0))\, dx_1,$$

hence

$$\left|\int_B x_1 \left(\frac{\partial \psi_2^*}{\partial x_1} dx_1 + \frac{\partial \psi_2^*}{\partial x_2} dx_2\right)\right| \leq 2\pi N \varepsilon_0.$$

For the last integral we have obviously the estimate (17) hence

$$|A_i| \geq 4\pi^2 N^2 - 4\pi N K_0 - 2\pi N \varepsilon_0 - 16\pi N K_0 \varepsilon_0,$$

and it follows that

$$|A_i| > |A_r|,$$

when N is great enough. But this is contradictory to the result derived in § 3. This completes the proof of Theorem 1.

Indleveret til selskabet den 13. januar 1950.
Færdig fra trykkeriet den 3. juni 1950.